durch die Felder benachbarter Atome erleidet, folgt für die resultierende mittlere Geschwindigkeit eines Teilchens nach einem Stoß: $\hat{v}_{\text{red}} = \hat{v} \cdot \varphi$.

Aus den Gesetzen des elastischen Stoßes für Kugeln der Massen m_0, m_1 ergibt sich für die Geschwindigkeit von m_0 nach dem Stoß mit

$$\beta = ((m_1 - m_0)/(m_1 + m_0))^2, \quad \varepsilon = (1/\beta - 1)\,(d/R_0)^2: \qquad v' = v_0 (\beta\,\varepsilon)^{1/2} ((1 + 1/\varepsilon) - \cos^2 \vartheta)^{1/2}; \quad \text{(A.6)}$$

weiterhin ist:

$$\cos\vartheta_{\min} = (1 - (R_0/d)^2 (\beta - (v_{\min}/v_0)^2)/(\beta - 1))^{1/2},$$
$$\cos\vartheta_{\max} = (1 - (R_0/d)^2)^{1/2}.$$
$$\hat{v} = \tfrac{1}{2} v_0 (\beta\,\varepsilon)^{1/2} (1 + 1/\varepsilon) \frac{U(\varepsilon, \cos\vartheta_{\max}) - U(\varepsilon, \cos\vartheta_{\min})}{\cos\vartheta_{\max} - \cos\vartheta_{\min}},$$

wo (A.7)

$$U(\varepsilon, \cos\vartheta) \equiv (X(1 - X))^{1/2} + \arctan(X/(1 - X))^{1/2}, \quad X \equiv \cos^2\vartheta/(1 + 1/\varepsilon).$$

On the Least Stable Mode in Cylindrical Systems

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The stability of a magnetodynamic equilibrium system for perturbations $\boldsymbol{\xi}$ depending on a parameter α is investigated. Growth rates are compared (rather than only the second order energy variations) without calculating them explicitly. The explicit dependence on α as wel as the implicit dependence on α through $\boldsymbol{\xi}$ is taken into account.

It is well known in magnetodynamics that a system is unstable or stable if the change in potential energy $\delta W(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ can or cannot be made negative [1]. Consider a displacement $\boldsymbol{\zeta}$ depending on a parameter α [α may be m^2 or k^2 of Eq. (1), e. g.], which for $\alpha = \alpha_1$, makes $\delta W(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ more negative than another displacement with $\alpha = \alpha_2$. In this case it often happens in the literature that the displacement with $\alpha = \alpha_1$ is called less stable or more dangerous than the one with $\alpha = \alpha_2$. In many cases this is not true at all. The expression *more or less stable* can only be used in a consistent way by comparing growth rates. Indeed, denoting the kinetic energy of the system for the perturbation $\boldsymbol{\zeta}$ by $K(\boldsymbol{\zeta}, \boldsymbol{\zeta})$, we have $\omega^2(\boldsymbol{\zeta}, \boldsymbol{\zeta}) = \delta W(\boldsymbol{\zeta}, \boldsymbol{\zeta})/K(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ where $K(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ is a function of α too; hence it is impossible to decide which displacement is the least stable from δW alone, or from a partly minimized δW, as is often done.

Another difficulty is that $\omega^2(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ is implicitly dependent on α through the eigenmode $\boldsymbol{\zeta}$ for which $\omega^2(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ is minimized by variation of $\boldsymbol{\zeta}$ (α kept fixed).

Since we may expect that the fastest growing mode will usually dominate in nature, it seems wortwhile to point out what can be asserted about the growth rates by simple arguments and without going into the solution of the eigenmode equations. This is done in the following sections for systems with rotational and cylindrical symmetry.

In section 1 the explicit dependence of $\omega^2(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ on α will be examined. The influence of the implicit dependence on α will be treated in section 2. The conclusions concerning the stability will be discussed in section 3.

1. Explicit Dependence of $\omega^2(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ on α

Consider the magnetodynamic stability of an equilibrium system with cylindrical and rotational symmetry (neither ϑ nor z dependence). A cylindrical ideally conducting plasma of density $\varrho(r)$ and pressure $p(r)$, is pervaded by a magnetic field $[0, B_\vartheta(r), B_z(r)]$ and a current $[0, j_\vartheta(r), j_z(r)]$. It is acted on by an external gravitational field $[-\mathrm{d}\varphi/\mathrm{d}r, 0, 0]$ which e. g. may simulate an acceleration. Normal mode solutions are of the form (cylindrical coordinates)

$$\zeta_n(r; m, k) \exp\{i(m\,\vartheta + k\,z + \omega\,t)\} + \text{c. c.} \qquad (1)$$

The eigenmode equation reduces to the two equations in r only:

$$-\varrho\,\omega_n^2 \boldsymbol{\zeta}_n = \boldsymbol{F}(\boldsymbol{\zeta}_n), \quad -\varrho\,\omega_n^2 \boldsymbol{\zeta}_n^* = \boldsymbol{F}^*(\boldsymbol{\zeta}_n^*). \qquad (2)$$

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[1] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. London **244**, 17 [1958].

The operators $\partial/\partial\vartheta$ and $\partial/\partial z$ in $\boldsymbol{F}$ are replaced by $i\,m$ and $i\,k$ respectively.

We introduce the functional

$$\omega^2(\boldsymbol{\zeta},\boldsymbol{\zeta}) = -\frac{\frac{1}{2}\pi\int\boldsymbol{\zeta}^*\cdot\boldsymbol{F}(\boldsymbol{\zeta})\,r\,\mathrm{d}r}{\frac{1}{2}\pi\int\varrho\,\boldsymbol{\zeta}^*\cdot\boldsymbol{\zeta}\,r\,\mathrm{d}r} = \frac{\delta W(\boldsymbol{\zeta},\boldsymbol{\zeta})}{K(\boldsymbol{\zeta},\boldsymbol{\zeta})}. \tag{3}$$

From now on, $\boldsymbol{\zeta}$ denotes, as well as $\boldsymbol{\zeta}_n$, a function of r, k and m. If $\boldsymbol{\zeta}$ is an eigenfunction $\boldsymbol{\zeta}_n$, $\omega^2(\boldsymbol{\zeta},\boldsymbol{\zeta})$ is constant and equals the eigenvalue $\omega_n{}^2$. The change in potential energy [Eqs. (2.6) and (2.7) of TAYLER [2]] is:

$$\begin{aligned}\delta W(\boldsymbol{\zeta},\boldsymbol{\zeta}) = \tfrac{1}{2}\pi\int r\,\mathrm{d}r\{&\tfrac{1}{2}\,(m\,B_\vartheta/r+k\,B_z)^2\,\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^* + \tfrac{1}{2}\,(B^2+\gamma\,p)\,\mathrm{div}\,\boldsymbol{\zeta}\,\mathrm{div}\,\boldsymbol{\zeta}^*\\ &-[B_\vartheta(B_\vartheta/r)^2+\tfrac{1}{2}\varrho'\,\varphi']\,\zeta_r\,\zeta_r^* - (2\,B_\vartheta^2/r+\varrho\,\varphi')\,\zeta_r\,\mathrm{div}\,\boldsymbol{\zeta}^*\\ &+i(m\,B_\vartheta/r+k\,B_z)\,(2\,B_\vartheta\,\zeta_\vartheta\,\zeta_r^*/r-\boldsymbol{B}\cdot\boldsymbol{\zeta}\,\mathrm{div}\,\boldsymbol{\zeta}^*)\}+\text{c. c.}\end{aligned} \tag{6}$$

A simple way to examine the explicit dependence of $\omega^2(\boldsymbol{\zeta},\boldsymbol{\zeta})$ on m^2 and k^2 is by introducing the new functions $\boldsymbol{\eta}$:

$$\eta_r=\zeta_r\,,\quad \eta_\vartheta=i\,m\,\zeta_\vartheta\,,\quad \eta_z=i\,k\,\zeta_z \tag{7}$$

for which div $\boldsymbol{\zeta}$ has no explicit dependence on m and k. K, expressed in $\boldsymbol{\eta}$, becomes a decreasing function of both $|m|$ and $|k|$, the trial functions $\boldsymbol{\eta}$ kept fixed.

We consider three cases, each containing two subcases, constituting 6 possibilities in total:

1. for helical magnetic fields we compare displacements with the same value of k/m, but with different values of k or m. Here the two subcases which occur by expressing $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})$ in k/m and m or in k/m and k, yield of course equivalent conclusions.
2. for longitudinal magnetic fields $(B_\vartheta=0)$ we compare displacements with different values of m while k is kept fixed, and vice-versa.
3. for transverse magnetic fields $(B_z=0)$: same analyses as case 2.

After careful inspection of $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})$ for the six possibilities mentioned, $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})$ turns out to be either one of the two standard forms I or II:

$$\omega_{\rm I}^2 = (a\,\alpha+b)/(c/\alpha+d) \tag{8.I}$$

$$\omega_{\rm II}^2 = (a/\alpha+b)/(c/\alpha+d) \tag{8.II}$$

where α stands for either m^2 or k^2. The coefficients a, b, c and d in (8.I) and (8.II) are functionals of $\boldsymbol{\eta}$ which may be different for each of the six possibilities. They depend on the parameters kept fixed also. a, c and d are always positive definite, while b may have either sign. The occurrence of the two forms I and II, acording to each case is shown in Table 2 which will be discussed in the third section.

Let us consider now the explicit dependence of the functionals $\omega_{\rm I}^2(\boldsymbol{\eta},\boldsymbol{\eta})$ and $\omega_{\rm II}^2(\boldsymbol{\eta},\boldsymbol{\eta})$ on the parameter α, while the trial functions $\boldsymbol{\eta}$ are kept fixed. The results of this discussion will be summarized in Table 1. A great simplification occurs because K (the denominator) is a decreasing function of α in both forms and is moreover always positive.

Form I. δW is an increasing function of α

If $\omega_{\rm I}^2>0$ then $\omega_{\rm I}^2$ increases with increasing α (since the numerator increases and the denominator decreases with increasing α, and both are positive).

If $\omega_{\rm I}^2<0$ then $\omega_{\rm I}^2$ may increase or decrease with α, according to the specific values of a, b, c and d. However it is easy to see that $\sigma_{\rm I}^2/\alpha$ decreases with α, $(\sigma_{\rm I}^2=-\omega_{\rm I}^2)$, so that $\sigma_{\rm I}^2$ *can decrease or increase, but cannot grow faster than α does.*

Form II. δW is a decreasing function of α

A similar argument shows that:

If $\omega_{\rm II}^2<0$ then $\sigma_{\rm II}^2$ increases with increasing α.

If $\omega_{\rm II}^2>0$ then $\omega_{\rm II}^2/\alpha$ decreases, so that $\omega_{\rm II}^2$ *can decrease or increase, but cannot grow faster than α does.*

Tables 1 and 2 are unaltered if the equilibrium is perturbed by incompressible, non-adiabatic displacements. Indeed, the authors [3] have shown that an energy principle technique can be applied with the same form of δW but with the constraint $\mathrm{div}\,\boldsymbol{\zeta}=0$.

	$\omega^2>0$	$\sigma^2=-\omega^2>0$
form I	$\omega_{\rm I}^2$ ↗	$\sigma_{\rm I}^2/\alpha$ ↘
form II	$\omega_{\rm II}^2/\alpha$ ↘	$\sigma_{\rm II}^2$ ↗

Table 1. Dependence of the two standard forms on α.

[2] R. J. TAYLER, J. Nucl. Energy Part C, 3, 266 [1961].

[3] J. G. KRÜGER and D. K. CALLEBAUT, Z. Naturforsch., to be published.

2. Implicit Dependence of $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})$ on α

2.1. Relation between total and partial derivative

This section is only based on the hermiticity of the operator $\boldsymbol{F}$, not upon its detailed structure.

In order to write the equation of motion (2) in a form which is applicable to the old variables $\boldsymbol{\zeta}$ as well as to new variables $\boldsymbol{\eta}$ we introduce a generalized form of Eq. (2):

$$-\varrho\,\omega_n^2 T\,\boldsymbol{\eta}_n=\boldsymbol{F}(\boldsymbol{\eta}_n). \tag{9}$$

If $\boldsymbol{\eta}_n=\boldsymbol{\zeta}_n$, T is the unit matrix; if $\boldsymbol{\eta}_n$ stands for the new variables (7), T is a diagonal matrix of which the non-zero elements are

$$T_{rr}=1, \quad T_{\vartheta\vartheta}=1/m^2, \quad T_{zz}=1/k^2. \tag{10}$$

Multiplying (9) with $\boldsymbol{\eta}_n^*$, integrating over the plasmavolume and replacing $\boldsymbol{\eta}_n$ by $\boldsymbol{\eta}$, we obtain instead of (3):

$$\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})=-\frac{\int\boldsymbol{\eta}^*\cdot\boldsymbol{F}(\boldsymbol{\eta})\,r\,\mathrm{d}r}{\int\varrho\,\boldsymbol{\eta}^*\cdot T\cdot\boldsymbol{\eta}\,r\,\mathrm{d}r}. \tag{11}$$

Taking into account the implicit dependence on α through $\boldsymbol{\eta}$, we obtain

$$\frac{\mathrm{d}\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})}{\mathrm{d}\alpha}=\frac{\partial\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})}{\partial\alpha} \tag{12}$$
$$-\frac{\int\left\{\frac{\partial\boldsymbol{\eta}^*}{\partial\alpha}\cdot[\boldsymbol{F}(\boldsymbol{\eta})+\varrho\,\omega^2 T\cdot\boldsymbol{\eta}]+\text{c. c.}\right\}r\,\mathrm{d}r+\Omega}{\int\varrho\,\boldsymbol{\eta}^*\cdot T\cdot\boldsymbol{\eta}\,r\,\mathrm{d}r}.$$

In $\partial\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})/\partial\alpha$ only $\boldsymbol{F}$ and T are derived, while $\boldsymbol{\eta}$ is kept fixed.

$$\Omega=\int\left[\boldsymbol{\eta}\cdot\boldsymbol{F}\left(\frac{\partial\boldsymbol{\eta}}{\partial\alpha}\right)-\frac{\partial\boldsymbol{\eta}}{\partial\alpha}\cdot\boldsymbol{F}(\boldsymbol{\eta})\right]r\,\mathrm{d}r. \tag{13}$$

We restrict the class of functions $\boldsymbol{\eta}$ to those functions satisfying the boundary conditions. As to $\partial\boldsymbol{\eta}/\partial\alpha$, two cases may occur: either $\partial\boldsymbol{\eta}/\partial\alpha$ satisfies or does not satisfy the boundary conditions.

Case a. $\partial\boldsymbol{\eta}/\partial\alpha$ satisfies the boundary conditions

This happens e. g. when the wall is fixed or at infinity, so that $\boldsymbol{\eta}$ and $\partial\boldsymbol{\eta}/\partial\alpha$ vanish at the boundary. Due to the selfadjointness of the operator $\boldsymbol{F}$, the integral Ω vanishes. If $\boldsymbol{\eta}$ is an eigensolution of (9), we have

$$\frac{\mathrm{d}\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})}{\mathrm{d}\alpha}=\frac{\partial\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})}{\partial\alpha}. \tag{14}$$

The equality of the total and partial derivatives can be understood as follows. ω^2 is an extremum with respect to all variations $\delta\boldsymbol{\eta}$ which satisfy the boundary conditions. Choose $\delta\boldsymbol{\eta}=(\partial\boldsymbol{\eta}/\partial\alpha)\,\mathrm{d}\alpha$. ω^2 will not be affected to first order by changing from $\boldsymbol{\eta}$ to $\boldsymbol{\eta}+(\partial\boldsymbol{\eta}/\partial\alpha)\,\mathrm{d}\alpha$, since ω^2 is extremum with respect to these functions. Hence there will be no contribution from $\boldsymbol{\eta}$ in the derivative of ω^2 with respect to α and $\mathrm{d}\omega^2/\mathrm{d}\alpha$ consists only of $\partial\omega^2/\partial\alpha$.

The equality of the total and partial derivatives in case a allows us to draw conclusions without knowing the eigensolutions. Indeed *if $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta};\alpha)$ is increasing* (respectively decreasing) *with α in an interval $\alpha_1<\alpha<\alpha_2$ independent of the trial function $\boldsymbol{\eta}$ kept fixed,* the function $\omega^2(\boldsymbol{\eta}_n,\boldsymbol{\eta}_n;\alpha)$ is of course increasing (resp. decreasing) with α in this interval, the eigensolution $\boldsymbol{\eta}_n$ kept fixed.

Hence, according to (14) *the eigenvalue $\omega^2(\alpha)$ itself is increasing* (resp. decreasing) *in the same interval.* More generally, if $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta};\alpha)$ has N extrema in α in an interval $\alpha_1<\alpha<\alpha_2$ independent of the trial function $\boldsymbol{\eta}$ kept fixed, the eigenvalue $\omega^2(\alpha)$ has also N extrema. Applying this reasoning to Table 1, we find that Table 1 is valid not only for $\omega^2(\boldsymbol{\eta},\boldsymbol{\eta})$ with $\boldsymbol{\eta}$ kept fixed, but also for *the eigenvalues* $\omega^2(\alpha)$ themselves.

Case b. $\partial\boldsymbol{\eta}/\partial\alpha$ does not satisfy the boundary conditions

For a plasma surrounded by vacuum it can be shown that Ω does not necessarily vanish. This can be understood by remarking that ω^2 will now in general be affected to first order by changing from $\boldsymbol{\eta}$ to $\boldsymbol{\eta}+(\partial\boldsymbol{\eta}/\partial\alpha)\,\mathrm{d}\alpha$:

$$\frac{\mathrm{d}\omega^2}{\mathrm{d}\alpha}=\frac{\partial\omega^2}{\partial\alpha}+\frac{\Omega}{K}. \tag{15}$$

2.2. Minimalization with respect to several parameters

So far as we have shown, Table 1 is valid for the eigenvalues $\omega^2(\alpha)$, comparing eigenvalues with different values of α but with all other parameters (discrete or not) kept fixed. Since the interest lies in the lowest value of ω^2 not only with respect to α but also with respect to the other parameters, we now look for the dependence on α of the minimum of ω^2 with respect to a set of parameters p_i $(i=1, 2,\ldots,l)$.

The extremum values of $\omega^2(\alpha,p_i)$ with respect to the l parameters p_i is obtained by the l equations

$$\partial\omega_{\mathrm{I}}^2(\alpha,p_i)/\partial p_i=0, \quad i=1,2,\ldots,l \tag{16}$$

which define the coordinates of the extrema, $p_i^0(\alpha)$, depending on α (and eventually on remaining parameters). Now

$$\frac{d\omega^2}{d\alpha} = \frac{\partial\omega^2}{\partial\alpha} + \sum_{i=1}^{l} \frac{\partial\omega^2}{\partial p_i} \frac{\partial p_i}{\partial\alpha} \tag{17}$$

which becomes, for the extrema defined by (16)

$$\frac{d\omega^2[\alpha, p_i^0(\alpha)]}{d\alpha} = \frac{\partial\omega^2[\alpha, p_i^0(\alpha)]}{\partial\alpha}. \tag{18}$$

We conclude from this equality: *Table 1 remains also valid for the minima of ω^2 satisfying (15).*

Example. If $\omega_{\rm I}^2[\alpha, p_i^0(\alpha)] > 0$ for a certain value of α, then $\omega_{\rm I}^2[\alpha, p_i^0(\alpha)]$ increases with α for all higher values of α. Note also that, if $\omega_{\rm I}^2[\alpha, p_i^0(\alpha)]$ is moreover the absolute minimum with respect to p_i, then, as $\omega_{\rm I}^2(\alpha, p_i) \geqq 0$ for all p_i, it follows from section 2.1. that $\omega_{\rm I}^2(\alpha, p_i)$ increases with α for all p_i and all higher values of α.

3. Conclusions Concerning Stability

As mentioned in the introduction, the expression *more or less stable* can only be used in a consistent way by comparing *growth rates.*

A perturbation with parameter $\alpha = \alpha_1$, is called less stable than one with $\alpha = \alpha_2$, if the first one contains at least one eigenmode for which $\omega^2(\alpha_1)$ is smaller than any $\omega^2(\alpha_2)$ corresponding to the eigenmodes of the second perturbation. According to this definition, the least stable displacements contain the fastest growing instability, if they are unstable. If they are stable, they correspond to the perturbations containing to smallest frequency. This definition can be applied with any number of other parameters kept fixed or not.

We now compare the stability of perturbations $\boldsymbol{\zeta}(r; m, k)$ with different values of k and m combining the results of Table 1 and Table 2.

3a. Dependence of ω^2 on m

3a.1. Helical fields. k/m fixed

For stable displacements, Tables 1 and 2 show that the eigenvalue ω^2 is an increasing function of $|m|$. Hence, if a displacement is stable for $|m| = |m_0|$, it is *more stable* for values higher than $|m_0|$.

From δW alone one could deduce only a less strong statement. Indeed, $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta})$ is an increasing function of $|m|$ and this allows only the conclusion: if a system is stable for $|m| = |m_0|$ it is *also* stable for $|m| > |m_0|$. This last statement can also be derived from a δW which is minimized with respect to ζ_ϑ and ζ_z, without normalization (NEWCOMB [4], Theorem 1).

From δW alone one might be inclined to expect that the lowest value of $|m|$ corresponds to the most dangerous perturbation. For the unstable modes however, nothing proves this, according to our analyses. On the contrary, in 3a.2. we meet an example in which $|m| = \infty$ is the most dangerous perturbation.

The conclusions of this paragraph are of course also valid for a purely longitudinal and a purely transverse magnetic field, which will be studied further in sections 3a.2. and 3a.3 with k or m fixed.

α	m^2	k^2
helical fields, k/m = constant	I	I
longitudinal fields ($B_\vartheta = 0$)	II	I
transverse fields ($B_z = 0$)	I	II

Table 2. Occurrence of the standard forms I and II.

3a.2. Purely longitudinal fields ($B_\vartheta = 0$). k fixed

For unstable displacements Tables 1 and 2 show that the eigenvalue ω^2 is a decreasing function of $|m|$. Hence, if a displacement is unstable for $|m| = |m_0|$, it is *more unstable* for values of $|m|$ higher than $|m_0|$.

From δW alone one could deduce only a less strong statement. Indeed, $\delta W(\boldsymbol{\eta}, \boldsymbol{\eta})$ is a decreasing function of $|m|$ and this allows only the conclusion: if a system is unstable for $|m| = |m_0|$, it is *also* unstable for $|m| > |m_0|$. Our analysis shows that $|m| \to \infty$ is the *most dangerous* displacement when $B_\vartheta = 0$, provided $\omega_{\rm II}^2$ is negative for at least one m. To discuss the latter restriction, we consider the minimized form $\delta W_{\rm F}$ of δW with respect to ζ_ϑ and ζ_z without normalization. Putting $B_\vartheta = 0$ in Eq. (2.15) or ref. [2], we obtain:

$$\delta W_{\rm F} = \frac{\pi}{2} \int r\, dr \left\{ \frac{k^2 B^2}{k^2 r^2 + m^2} [(r \zeta_r)']^2 + \left[k^2 B^2 - \varrho' \varphi' - \frac{(\varrho \varphi')^2}{\gamma p} \right] \zeta_r^2 \right\}. \tag{19}$$

[4] W. NEWCOMB, Ann. Phys. 10, 212 [1960].

In the presence of an external potential, it is always possible to make δW_F negative for sufficiently small k and thus to make $|m| \to \infty$ the most unstable displacement.

On the other hand, choose k sufficiently large so that δW_F is positive for all m; then $\omega_{II}^2 > 0$ for all m and ω_{II}^2 can increase as well as decrease with increasing m, according to Table 1. [Incidentally, in the limit $k \to 0$, δW as well as δW_F become independent of m; this illustrates a particular case of (8.II) in which $a = 0$; from the requirement that the trivial solution $\boldsymbol{\eta} = 0$ is excluded, one can show that $c \neq 0$. The discussion of Table 1 then remains valid. Thus for $k \to 0$, the most unstable displacement is $|m| \to \infty$ also.]

The fact that $|m| \to \infty$ may be the most unstable displacement under certain circumstances, is rather surprising in magnetodynamics. The situation may occur for a spiral arm, which is idealized as an infinitely long cylinder and where, in first approximation, gravitation acts as an external potential. AMANO et al.[5] claimed for this model on the basis of (19) alone [which is equivalent with their Eq. (2.3)] that $|m| \to \infty$ is most dangerous. Our analysis proves this statement. We make further the following annotations. As this situation occurs when $B_\vartheta = 0$ (or B_ϑ small) there is no transverse field to prohibit the "rippling" of the cylindrical tube, parallel to its axis. The asymptotic value $\sigma_{II}^2(m \to \infty)$ will be approached very well for a reasonable large $m = M$ ($M = 10$, say), as is suggested by (19). Thus, although $\sigma_{II}^2(m \to \infty) > \sigma_{II}^2(m = M)$, the relative difference in growth rate is fairly unimportant. All large values are then nearly equally able to dominate[6] and what perturbations will actually dominate becomes dependent on other factors as e. g. the probability of occurrence of a particular initial perturbation.

[5] T. AMANO, M. SATO, and Y. TERASHIMA, Progr. Theor. Phys. Suppl. **31**, 131 [1964].

3a.3. Purely transverse fields ($B_z = 0$). k fixed

This case is similar to 3a.1. If the system is stable for $|m| = |m_0|$ it is *more stable* for $|m| > |m_0|$. This statement is then valid for k/m fixed or k fixed provided $B_z = 0$. Nothing can be said in general for unstable displacements.

3b. Dependence of ω^2 on k

Similar arguments hold for the dependence on k as for the dependence on m. It may be sufficient to state the results.

3b.1. Helical fields. k/m fixed

If the system is stable for $|k| = |k_0|$, it is *more stable* for $|k| > |k_0|$. This conclusion is equivalent to the conclusion under 3a.1. Both conclusions express the fact that, in a (k, m)plane, $\omega^2(k, m)$ decreases when approaching the origin, along a straight line through the origin.

3b.2. Longitudinal fields. m fixed

If the system is stable for $|k| = |k_0|$ it is *more stable* for $|k| > |k_0|$. This statement is thus valid for k/m or k fixed provided $B_\vartheta = 0$.

3b.3. Transverse fields. m fixed

If the system is unstable for $|k| = |k_0|$ it is *more unstable* for $|k| > |k_0|$. This demonstrates that $|k| \to \infty$ is the most dangerous perturbation provided ω^2 can be made negative for at least one k.

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[6] An article on the importance of the bandwidth around the maximum mode of instability will appear later.